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STRESS WAVE PROPAGATION IN A TWO-LAYERED CYLINDER WITH INITIAL INTERFACE PRESSURE

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Abstract—This paper presents an elastodynamic solution for stress wave propagation in a two-layered cylinder with initial interface pressure. The initial interface pressure in a layered cylinder is expressed as the initial condition of an elastodynamic equilibrium equation. Elastodynamic equations of each separate hollow cylinder fulfilling the initial conditions are solved by means of a finite Hankel transform and Laplace transform. The boundary conditions at the internal–external surface and the interface continuity conditions are used to determine the unknown constants involved in solution. Thus, an exact solution for stress wave propagation in a two-layered cylinder with initial interface pressure is obtained.

INTRODUCTION

It is well known that the general solution for stress wave propagation in a hollow circular cylinder is very useful in engineering applications, such as nondestructive evaluation of material properties, flaw detection and determination of resonances. In the past, a number of analytical solutions for stress wave propagation in a single solid structure made of one medium have been presented by Torvik (1967), Achenbach and Fang (1970) and Pao (1983). However, the stress wave propagation in a layered structure subjected to shocking loading is still valuable to widely technological applications and is a more complex problem. So far, the subject has not been studied as extensively as it should have been. During the past few years, vibrations of layered shells have been studied by Yu (1960), Chu (1961), Bieniek and Freudenthal (1962), and Jones and Whittier (1966). But, their solving method based on a Timoshenko-type theory was limited to layered shell-type structures and was only used to calculate vibrations in a layered structure. Recently, Wang and Gong (1992) presented an elastodynamic solution for a layered cylinder without considering the initial interface pressure.

In this paper, an exact solution for stress wave propagation in a two-layered cylinder with initial interface pressure is derived and solved by means of the work of Wang and Gong (1991) which presents an effectively finite Hankel transform method. At first, the initial displacement and stress in a layered cylinder produced by initial interface pressure are expressed as the initial conditions of the elastodynamic equation. Then, governing equations for each layer of the cylinders are derived and solved. The solution comprises quasi-static solution with unknown constants and a dynamic solution meeting the homogeneous boundary conditions of each separate layer. From the boundary conditions and the interface continuity conditions of a layered cylinder, we can easily determine the unknown constants involved in the solution. Therefore, an exact solution for stress wave propagation in a two-layered cylinder with initial interface pressure is obtained. Through two examples, we demonstrate that the present method is simple, effective and correct.

THE INITIAL STATE OF A TWO-LAYERED CYLINDER

Consider a two-layered cylinder with initial interface pressure, composed of two hollow cylinders by means of a heat-assembling method. The initial interface pressure is caused by

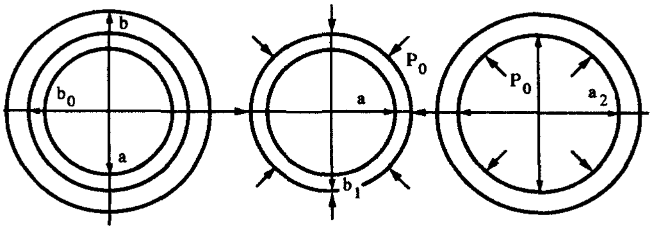

Fig. 1. The geometry of a two-layered cylinder.

the assembling pressure. The geometry of a two-layered cylinder is shown in Fig. 1, where a is specified as the internal radius of the first layer cylinder and b is specified as the external radius of the second layer cylinder. The interface radius of the layered cylinder is shown as b_0. Before two hollow cylinders are assembled, there is $b_1 - a_2 = \delta_0$, where b_1 and a_2 are specified as the external radius of the first layer cylinder and the internal radius of the second layer cylinder, respectively.

Applying static elasticity theory, we can describe the assembling pressure (the interface pressure) as

$$P_0 = \frac{\delta_0}{UB2 - UB1}, \tag{1a}$$

where

$$UB1 = \frac{b_0^3}{2(\lambda_1+\mu_1)(a^2-b_0^2)} + \frac{a^2 b_0^2}{2\mu_1(a^2-b_0^2)}, \tag{1b}$$

$$UB2 = \frac{b_0^3}{2(\lambda_2+\mu_2)(b^2-b_0^2)} + \frac{b^2 b_0^2}{2\mu_2(b^2-b_0^2)}, \tag{1c}$$

λ_j, μ_j are specified as the Lamés constants.

The initial displacement of the layered cylinder can be expressed as

$$U_{os}^1(r) = \left[\frac{b_0^2}{2(\lambda_1+\mu_1)(a^2-b_0^2)} r + \frac{a^2 b_0^2}{2\mu_1(a^2-b_0^2)}\frac{1}{r}\right] p_0, \quad (a \leqslant r \leqslant b_0), \tag{2a}$$

$$U_{os}^2(r) = \left[\frac{b_0^2}{2(\lambda_2+\mu_2)(b^2-b_0^2)} r + \frac{b^2 b_0^2}{2\mu_2(b^2-b_0^2)}\frac{1}{r}\right] p_0, \quad (b_0 \leqslant r \leqslant b). \tag{2b}$$

ELASTODYNAMIC BASIC EQUATION AND SOLUTION

Consider a layered cylinder subjected to a uniformly distributed, time-dependent interior pressure $\psi(t)$. The present problem is considered to be axially symmetric and a state of plane strain. The elastodynamic equations of the jth layered cylinder are found to be

$$\frac{\partial^2 U^j(r,t)}{\partial r^2} + \frac{1}{r}\frac{\partial U^j(r,t)}{\partial r} - \frac{U^j(r,t)}{r^2} = \frac{1}{V_j^2}\frac{\partial^2 U^j(r,t)}{\partial t^2}, \quad a_j \leqslant r \leqslant b_j, \quad t \geqslant 0^+, \quad (j = 1,2), \tag{3}$$

where $V_j = \sqrt{(\lambda_j + 2\mu_j)/\rho_j}$ is the wave speed and ρ_j is the mass density.

Considering that the layered cylinder with interface pressure is initially at rest, the initial condition of eqn (3) is expressed as

$$U^j(r,0) = U^j_{os}(r), \quad \frac{\partial U^j(r,0)}{\partial t} = 0. \tag{4}$$

It has been proposed that the general solution of the jth elastodynamic equation (3) could be described in the following form:

$$U^j(r,t) = U^j_s(r,t) + U^j_d(r,t), \tag{5}$$

where $U^j_s(r,t)$ is a quasi-static solution of eqn (3), which satisfies the following quasi-static equation and initial condition:

$$\frac{\partial^2 U^j_s}{\partial r^2} + \frac{1}{r}\frac{\partial U^j_s}{\partial r} - \frac{U^j_s}{r^2} = 0, \tag{6}$$

$$U^j_s(r,0) = U^j_{os}(r), \quad \frac{\partial U^j_s(r,0)}{\partial t} = 0. \tag{7}$$

The quasi-static solution of eqn (6) is derived as

$$U^j_s(r,t) = U^j_{st}(r,t) + U^j_{os}(r), \tag{8a}$$

where

$$U^j_{st}(r,t) = \phi^j(r)\cdot\psi(t), \quad \phi^j(r) = D^j_1 r + D^j_2 \frac{1}{r}. \tag{8b,c}$$

$\psi(t)$ is defined as the function of dynamic loading. The unknown constants D^j_1 and D^j_2 will be determined by the boundary condition and the interface continuity condition of the layered cylinder.

Substituting eqn (8) into (3) and utilizing eqns (4), (5), we have

$$\frac{\partial^2 U^j_d}{\partial r^2} + \frac{1}{r}\frac{\partial U^j_d}{\partial r} - \frac{U^j_d}{r^2} = \frac{1}{V^2_j}\left[\frac{\partial^2 U^j_d}{\partial t^2} + \frac{\partial^2 U^j_{st}}{\partial t^2}\right], \tag{9a}$$

$$U^j_d(r,0) = \frac{\partial U^j_d(r,0)}{\partial t} = 0, \tag{9b}$$

$$\sigma_r|_{r=a_j,b_j} = \left[(\lambda_j + 2\mu_j)\frac{\partial U^j_d}{\partial r} + \frac{\lambda_j}{r}U^j_d\right]_{r=a_j,b_j} = 0, \tag{9c,d}$$

where $U^j_d(r,t)$ is the solution of the inhomogeneous equation (9a) with zero initial conditions (9b) and the homogeneous boundary conditions (9c, d). U^j_{st} in eqn (9a) is the known function as shown in eqn (8). If we define $\bar{U}^j_d(\xi^j_i, t)$ as the finite Hankel transform of $U^j_d(r,t)$, we will have

$$\bar{U}^j_d(\xi^j_i, t) = H[U^j_d(r,t)] = \int_{a_j}^{b_j} rU^j_d(r,t)C_1(\xi^j_i r)\,\mathrm{d}r. \tag{10}$$

Then, by making use of the inverse of the transform, we have

$$U^j_d(r,t) = \sum_i \frac{\bar{U}^j_d(\xi^j_i, t)}{F(\xi^j_i)} C_1(\xi^j_i r), \tag{11a}$$

where

$$F(\xi_i^j) = \int_{a_j}^{b_j} r[C_1(\xi_i^j r)^2]\,\mathrm{d}r, \tag{11b}$$

$$C_1(\xi_i^j r) = J_1(\xi_i^j r)Y_a - Y_1(\xi_i^j r)J_a. \tag{11c}$$

$J_1(\xi_i^j r)$ and $Y_1(\xi_i^j r)$ are the first and second kind of first order Bessel functions, respectively. ξ_i^j are expressed as the positive roots of the following eigenequation by Wang and Gong (1991):

$$Y_a J_b - Y_b J_a = 0, \tag{12a}$$

where

$$Y_a = \xi_i Y_1'(\xi_i a) + h_1 Y_1(\xi_i a), \quad J_a = \xi_i J_1'(\xi_i a) + h_1 J_1(\xi_i a),$$
$$Y_b = \xi_i Y_1'(\xi_i b) + h_2 Y_1(\xi_i b), \quad J_b = \xi_i J_1'(\xi_i b) + h_2 J_1(\xi_i b),$$
$$h_1 = \frac{\lambda_j}{a_j(\lambda_j + 2\mu_j)}, \qquad h_2 = \frac{\lambda_j}{b_j(\lambda_j + 2\mu_j)}. \tag{12b–g}$$

Performing the finite Hankel transform of the inhomogeneous equation (9a), we have

$$\frac{2}{\pi}\frac{J_a^j}{J_b^j}[U_\mathrm{d}^{j\prime}(b_j) + h_2^j U_\mathrm{d}^j(b_j)] - \frac{2}{\pi}[U_\mathrm{d}^{j\prime}(a_j) + h_1^j U_\mathrm{d}^j(a_j)] - (\xi_i^j)^2 \bar{U}_\mathrm{d}^j(\xi_i^j) = \frac{1}{V_j^2}\left[\frac{\mathrm{d}^2\bar{U}_\mathrm{d}^j}{\mathrm{d}t^2} + \frac{\mathrm{d}^2\bar{U}_\mathrm{st}^j}{\mathrm{d}t^2}\right]. \tag{13}$$

Because $U_\mathrm{d}^j(r, t)$ satisfies the homogeneous boundary conditions (9c, d), the first two terms on the left-hand side of eqn (13) must equal zero. Thus, eqn (13) is reduced to

$$-(\xi_i^j)^2 \bar{U}_\mathrm{d}^j(\xi_i^j, t) = \frac{1}{V_j^2}\left[\frac{\mathrm{d}^2\bar{U}_\mathrm{d}^j}{\mathrm{d}t^2} + \frac{\mathrm{d}^2 U_\mathrm{st}^j}{\mathrm{d}t^2}\right]. \tag{14}$$

Performing the Laplace transform of eqn (14) and utilizing the initial conditions (11b), we have

$$\bar{U}_\mathrm{d}^{j*} = -\bar{U}_\mathrm{st}^{j*} + \frac{(\xi_i^j)^2 V_j^2}{(\xi_i^j V_j)^2 + p^2}\bar{U}_\mathrm{st}^{j*}, \tag{15}$$

where p is the parameter of the Laplace transform. The Laplace inverse transform of eqn (15) gives

$$\bar{U}_\mathrm{d}^j(\xi_i^j, t) = D_1^j \sum_i \left[\frac{RI_1^j(\xi_i^j, t)}{F^j(\xi_i^j)} C_1(\xi_i^j r)\right] + D_2^j \sum_i \left[\frac{RI_2^j(\xi_i^j, t)}{F^j(\xi_i^j)} C_1(\xi_i^j r)\right],$$

where

$$RI_1^j = \{H[r]\}\cdot I^j, \quad RI_2^j = \left\{H\left[\frac{1}{r}\right]\right\}\cdot I^j,$$

$$I^j = -\psi(t) + \xi_i^j V_j \int_0^t \psi(\tau)\sin[\xi_i^j V_j(t-\tau)]\,\mathrm{d}\tau. \tag{16a–e}$$

By substituting eqns (16) and (8) into (5), we can express the general solution of basic eqns (3) and (4) as

$$U^j(r,t) = D_1^j\left\{r\psi(t) + \sum_i\left[\frac{RI_1^j(\xi_i^j,t)}{F^j(\xi_i^j)}C_1(\xi_i^j r)\right]\right\}$$

$$+D_2^j\left\{\frac{1}{r}\psi(t) + \sum_i\left[\frac{RI_2^j(\xi_i^j,t)}{F^j(\xi_i^j)}C_1(\xi_i^j r)\right]\right\} + U_{os}^j(r). \quad (17)$$

By making use of Hooke's law, geometry relations and eqn (17) we easily gain the expressions of the dynamic stresses:

$$\sigma_r^j = \left[(\lambda_j+2\mu_j)\frac{\partial U^j(r,t)}{\partial r} + \frac{\lambda_j}{r}U^j(r,t)\right], \quad (18a)$$

$$\sigma_\theta^j = \left[\lambda_j\frac{\partial U^j(r,t)}{\partial r} + \frac{(\lambda_j+2\mu_j)}{r}U^j(r,t)\right], \quad (18b)$$

$$\sigma_z^j = v^j(\sigma_r^j+\sigma_\theta^j), \quad (j = 1,2). \quad (18c)$$

The above expressions have four unknown constants D_1^j and D_2^j $(j = 1,2)$ which can be determined from the boundary conditions and the interface continuity conditions of a two-layered cylinder.

BOUNDARY AND CONTINUITY CONDITIONS

Supposing that there is a uniformly distributed, time-dependent interior pressure $\psi(t)$, we can describe the boundary conditions and the continuity conditions of a two-layered cylinder as

$$\sigma_r^1(a,t) = \psi(t), \qquad \sigma_r^2(b,t) = 0,$$

$$U^1(r,t)|_{r=b_0} = U^2(r,t)|_{r=b_0}, \quad \sigma_r^1(r,t)|_{r=b_0} = \sigma_r^2(r,t)|_{r=b_0}. \quad (19a\text{–}d)$$

Substituting the expressions of stress and displacement (17), (18) into the boundary and continuity conditions (19), we have the following equations which are used to determine the unknown constants D_1^j and D_2^j $(j = 1,2)$:

$$\left[2(\lambda_1+\mu_1)D_1^1 - \frac{2\mu_1}{a^2}D_2^1\right]\psi(t) + \left[(\lambda_1+2\mu_1)\frac{\mathrm{d}U_{os}^1(a)}{\mathrm{d}r} + \frac{\lambda_1}{a}U_{os}^1(a)\right] = \psi(t), \quad (20a)$$

$$\left[2(\lambda_2+\mu_2)D_1^2 - \frac{2\mu_2}{b^2}D_2^2\right]\psi(t) + \left[(\lambda_2+2\mu_2)\frac{\mathrm{d}U_{os}^2(b)}{\mathrm{d}r} + \frac{\lambda_2}{b}U_{os}^2(b)\right] = 0, \quad (20b)$$

$$2(\lambda_1+\mu_1)D_1^1 - \frac{2\mu_1}{b_0^2}D_2^1 = 2(\lambda_2+\mu_2)D_1^2 - \frac{2\mu_2}{b_0^2}D_2^2, \quad (20c)$$

$$D_1^1\left\{b_0\psi(t) + \sum_i\left[\frac{RI_1^1}{F^1}C_1(\xi_i^1 b_0)\right]\right\} + D_2^1\left\{\frac{1}{b_0}\psi(t) + \sum_i\left[\frac{RI_2^1}{F^1}C_1(\xi_i^1 b_0)\right]\right\} + U_{os}^1(b_0)$$

$$= D_1^2\left\{b_0\psi(t) + \sum_i\left[\frac{RI_1^2}{F^2}C_1(\xi_i^2 b_0)\right]\right\} + D_2^2\left\{\frac{1}{b_0}\psi(t) + \sum_i\left[\frac{RI_2^2}{F^2}C_1(\xi_i^2 b_0)\right]\right\} + U_{os}^2(b_0).$$

(20d)

Because there are four equations in (20), the four unknown constants D_1^j, D_2^j, $(j = 1,2)$

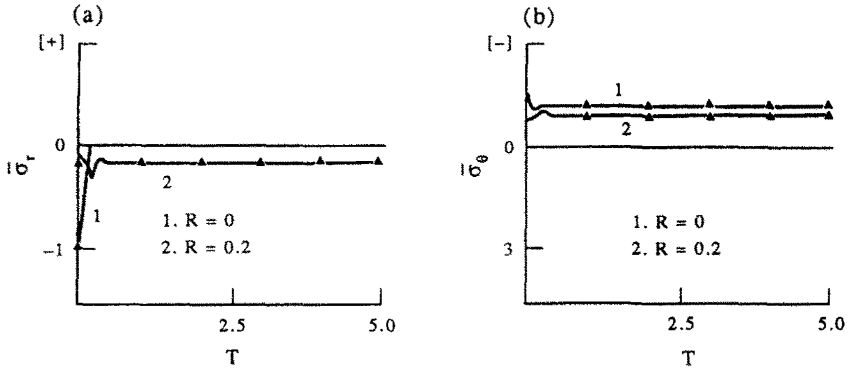

Fig. 2. Histories of dynamic stresses without the effect of reflected waves. $\bar{\sigma}_i = \sigma_i/\sigma_0$, $(b-a)/a = 10$, $(b_0-a)/a = 2$, $R = (r-a)/a$, $T = V \cdot t/a$.

can be easily determined. Thus, the dynamic displacements and the dynamic stresses in a two-layered cylinder with initial interface pressure are obtained exactly.

EXAMPLES AND DISCUSSIONS

Considering that the internal boundary of a two-layered cylinder is subjected to a suddenly exponential decay loading, we have

$$\psi(t) = \begin{cases} 0 & t < 0, \\ -\sigma_0 \, e^{-\alpha t} & t_0 > t \geqslant 0^+, \\ 0 & t \geqslant t_0, \end{cases} \tag{21}$$

where σ_0 shows the amplitude of the dynamic loading and $\alpha > 0$ is a factor to indicate the

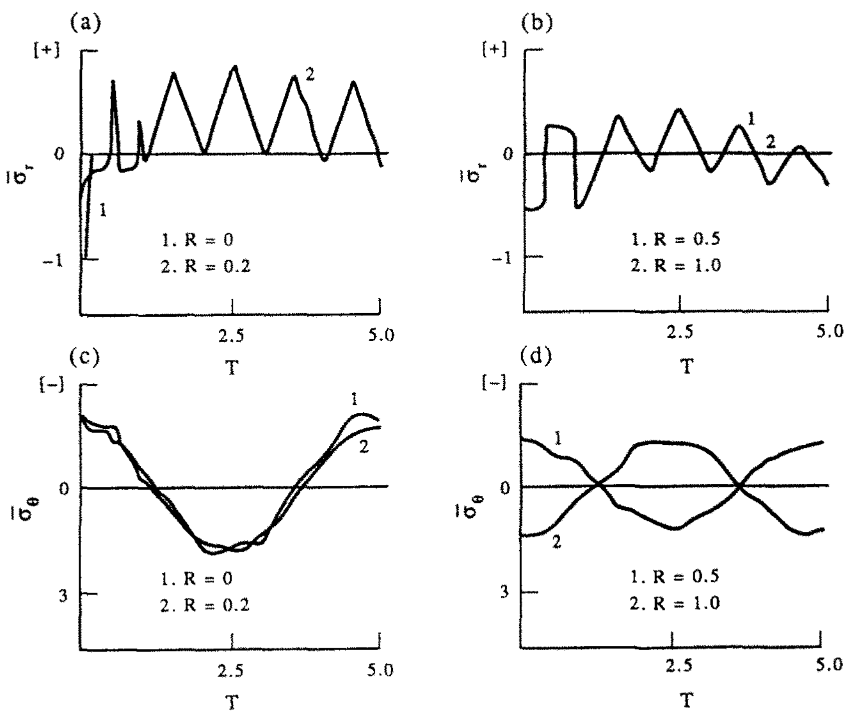

Fig. 3. Histories of dynamic stresses: $\bar{\sigma}_i = \sigma_i/\sigma_0$, $(b-a)/a = 1$, $(b_0-a)/a = 0.5$, $R = (r-a)/a$, $T = \dfrac{V \cdot t}{2(b-a)}$.

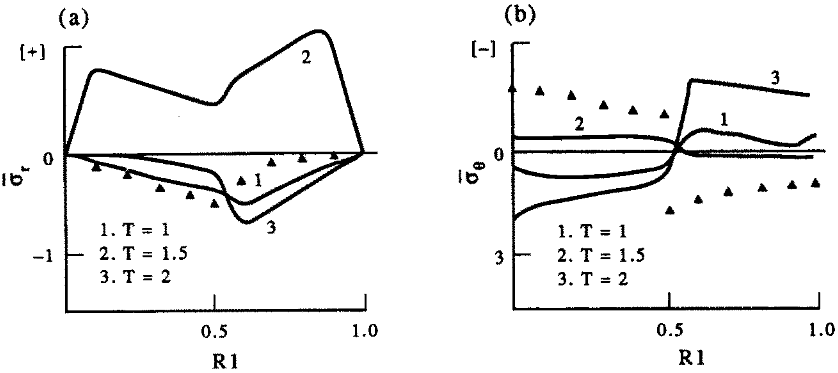

Fig. 4. Distributions of dynamic stresses: $\bar{\sigma}_i = \sigma_i/\sigma_0$, $(b-a)/a = 1$, $(b_0-a)/a = 0.5$, $R1 = (r-a)/(b-a)$, $T = \frac{V \cdot t}{2(b-a)}$.

rapidity of the decay. If α is very large, ($\alpha = 350{,}000$), a suddenly exponential decay loading will be approximately equal to a shocking load. In order to consider the initial interface pressure having influence on dynamic stresses, we only calculate the two-layered cylinder made of the same material. The material constants are $\lambda = \mu = 80$ GP_a, and $V = 5000$ m s^{-1}, respectively. The results of the numerical evaluation of stress wave propagation are illustrated in Figs 2–4. In these figures we calculate two structures with $(b-a)/a = 10$, $(b_0-a)/a = 2$, and $(b-a)/a = 1$, $(b_0-a)/a = 0.5$. The initial interface pressure of two structures is $p_0/\sigma_0 = 0.5$; ▲ expresses the initial stress at various locations.

The histories and distributions of the dynamic stresses for $\alpha = 350{,}000$ and $(b-a)/a = 10$ are shown in Fig. 2. From curve 1 in Fig. 2(a) we can see that sudden decay in interior pressure is approximately equivalent to a shocking load. Curve 2 in Fig. 2(a) and the curves in Fig. 2(b) have clearly shown the features of the cylindrical waves propagating in the cylinder with the initial stress field. The stress is approximately equal to the initial stress before the arrival of the wavefront. A discontinuity at the wavefront and oscillations behind the wavefront respond to the shocking load. It is observed that the dynamic stresses approach the initial stresses when T is large. The propagation of the wavefront in the cylinder with the initial stress fields is similar to that in the cylinder with the zero initial condition (Wang and Gong, 1991). It is concluded that the solution is correct.

Because of the effects of reflected waves we observe that the histories of dynamic stresses in a cylinder with the initial stress fields and $(b-a)/a = 1$ in Fig. 3 are different from those in Fig. 2. In this case, when the dynamic load disappeared, the oscillations were still accompanied by the stress waves propagating between the interior and exterior boundary, where the reflected waves are produced successively upon the arrival of the incident waves. It is worthwhile noting that the oscillations of compression stress waves in the initial stress field will sometimes cause the radial tension stress.

Figure 4 shows that the distributions of dynamic stresses along the radius r vary rapidly with time T. The distributions of radial and tangential stresses are shown in Fig. 4. From Fig. 4, we observe that the distributions of the dynamic stresses at the time the wavefront proceeds from the internal surface to the outer is obviously different from that at the time of the wavefront proceeding from the external surface to the inner. The phenomenon is mainly in relation to the interference of the initial stress field and is very interesting.

Finally, we conclude that the major accomplishment of this study has been in gaining a better solution method for the stress wave propagation in a two-layered cylinder with initial interface pressure. The analysis may be applied to a wide range of structure analyses in consideration of dynamic effects.

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